

Heat and Mass transfer

Velocity and Thermal Boundary Layers

The general form of energy equation:

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + k \nabla^2 T + \mu \Phi + \dot{S}_{th}$$

This is the general form of energy equation.

Special Case: For liquids, we can directly write :

$$\rho c \frac{DT}{Dt} = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right] + \mu \Phi + \dot{S}_{th}$$

For liquids, $c_p = c_v = c$ and $p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$. In the case of liquids, the viscous dissipation term will have to be modified by invoking the incompressibility condition.

Important Dimensionless Numbers

In order to nondimensionalize, we define

$$u^* = \frac{u}{U_\infty}, \quad v^* = \frac{v}{U_\infty}, \quad w^* = \frac{w}{U_\infty}, \quad \theta = \frac{T - T_\infty}{T_w - T_\infty},$$

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L}, \quad p^* = \frac{p}{\rho U_\infty^2}, \quad t^* = t \frac{U_\infty}{L}$$

Assumption of $\dot{S}_{th}=0$ produces,

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = \left(\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} \right)$$

$$+ k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \mu \Phi$$

The time derivatives will vanish for the steady state situation. In any case,

$$\frac{\partial \theta}{\partial t^*} + u^* \frac{\partial \theta}{\partial x^*} + v^* \frac{\partial \theta}{\partial y^*} + w^* \frac{\partial \theta}{\partial z^*} = \frac{U_\infty^2}{c_p(T_w - T_\infty)} \left(\frac{\partial p^*}{\partial t^*} + u^* \frac{\partial p^*}{\partial x^*} + v^* \frac{\partial p^*}{\partial y^*} + w^* \frac{\partial p^*}{\partial z^*} \right)$$

$$+ \frac{k}{\rho c_p U_\infty L} \left(\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}} + \frac{\partial^2 \theta}{\partial z^{*2}} \right) + \frac{\mu U_\infty^2}{L \rho c_p (T_w - T_\infty) U_\infty} \bar{\Phi}$$

$$\frac{\partial \theta}{\partial t^*} + u^* \frac{\partial \theta}{\partial x^*} + v^* \frac{\partial \theta}{\partial y^*} + w^* \frac{\partial \theta}{\partial z^*} = Ec \left(\frac{\partial p^*}{\partial t^*} + u^* \frac{\partial p^*}{\partial x^*} + v^* \frac{\partial p^*}{\partial y^*} + w^* \frac{\partial p^*}{\partial z^*} \right) + \frac{1}{Re.Pr} \left(\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}} + \frac{\partial^2 \theta}{\partial z^{*2}} \right) + \frac{Ec}{Re} \bar{\Phi}$$

$$Ec = \frac{U_\infty^2}{c_p(T_w - T_\infty)}, \quad Pr = \frac{\mu c_p}{k}, \quad Re = \rho \frac{U_\infty L}{\mu}$$

Let us look at the term Ec (Eckert number)

$$Ec = \frac{U_\infty^2}{c_p(T_w - T_\infty)} = \frac{\text{Kinetic energy of the flow}}{\text{Boundary layer enthalpy difference}}$$

$$\begin{aligned} \frac{1}{Ec} &= \frac{c_p(T_w - T_\infty)}{U_\infty^2} = \frac{c_p T_\infty}{U_\infty^2} \left(\frac{T_w}{T_\infty} - 1 \right) \\ &= \frac{c_p}{\gamma(c_p - c_v)} \frac{a^2}{U_\infty^2} \left(\frac{T_w}{T_\infty} - 1 \right) = \frac{1}{(\gamma - 1)} \frac{1}{M^2} \left(\frac{T_w}{T_\infty} - 1 \right) \end{aligned}$$

So,

$$Ec = \frac{(\gamma - 1)M^2}{[(T_w/T_\infty) - 1]}$$

where $M = (\text{fluid velocity}) / (\text{local sound speed})$ and a is the local sound speed, and M is known as Mach Number.

For low Mach number situation, energy equation in nondimensional form is

$$\frac{\partial \theta}{\partial t^*} + u^* \frac{\partial \theta}{\partial x^*} + v^* \frac{\partial \theta}{\partial y^*} + w^* \frac{\partial \theta}{\partial z^*} = \frac{1}{Re.Pr} \left[\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}} + \frac{\partial^2 \theta}{\partial z^{*2}} \right]$$

Dimensional form will be

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right]$$

For forced convection, the variables can be presented as

$$\theta = \theta(u, v, x, y, Re, Pr, T)$$

$$q_w = -k_f \frac{\partial T}{\partial n} = h(T_w - T_\infty)$$

Where, n is the dimension in the direction normal to the surface.

or,

$$h = \frac{-k_f \frac{\partial T}{\partial n}}{(T_w - T_\infty)} = \frac{-k_f \frac{(T_w - T_\infty)}{L} \frac{\partial \theta}{\partial \bar{n}}}{T_w - T_\infty}$$

where $\bar{n} = \frac{n}{L}$

or,

$$\left(\frac{hL}{k} \right) = Nu = - \left(\frac{\partial \theta}{\partial \bar{n}} \right)$$

Therefore, Nusselt number, Nu is the nondimensional temperature gradient at the surface on which heat transfer takes place.

Boundary Layers

Velocity Boundary Layer

Transition of zero velocity at the surface to the freestream velocity U_∞ takes place through a very thin layer δ . With the increase in y from the surface, the x velocity component, u , must increase until it approaches U_∞ . The quantity δ is termed as boundary layer thickness (Figure 1.3) and it is formally defined as the value of y for which $u = 0.99 U_\infty$. The flow field has two regions:

- (a) The region where $(\partial u / \partial y)$ and consequently shear stress is significant.
- (b) The region where $(\partial u / \partial y)$ and shear stress is negligible.

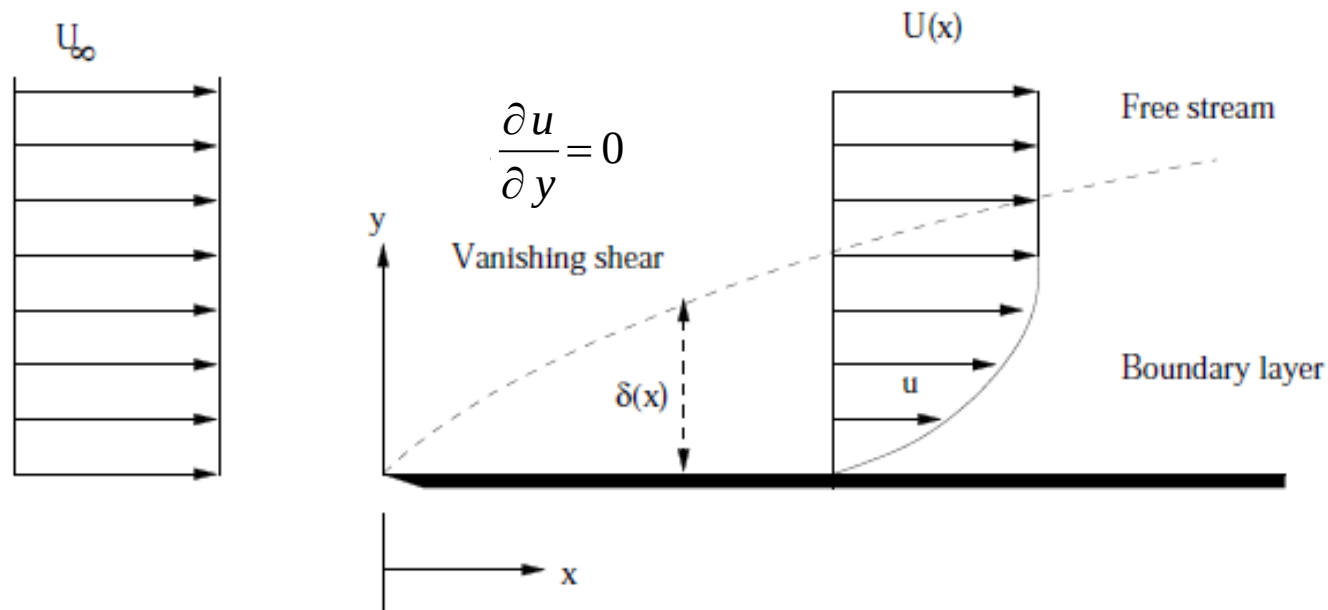


Figure 1.3: Velocity Boundary Layer on a Flat Plate

Here $\partial u / \partial y$ determines the local friction coefficient, and $\delta_x = 5.0x / \sqrt{Re_x}$

$$C_f = \tau_w / \frac{1}{2} \rho U_\infty^2$$

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0}$$

The slope, (du/dy) changes with x as shown in Figure 1.4.

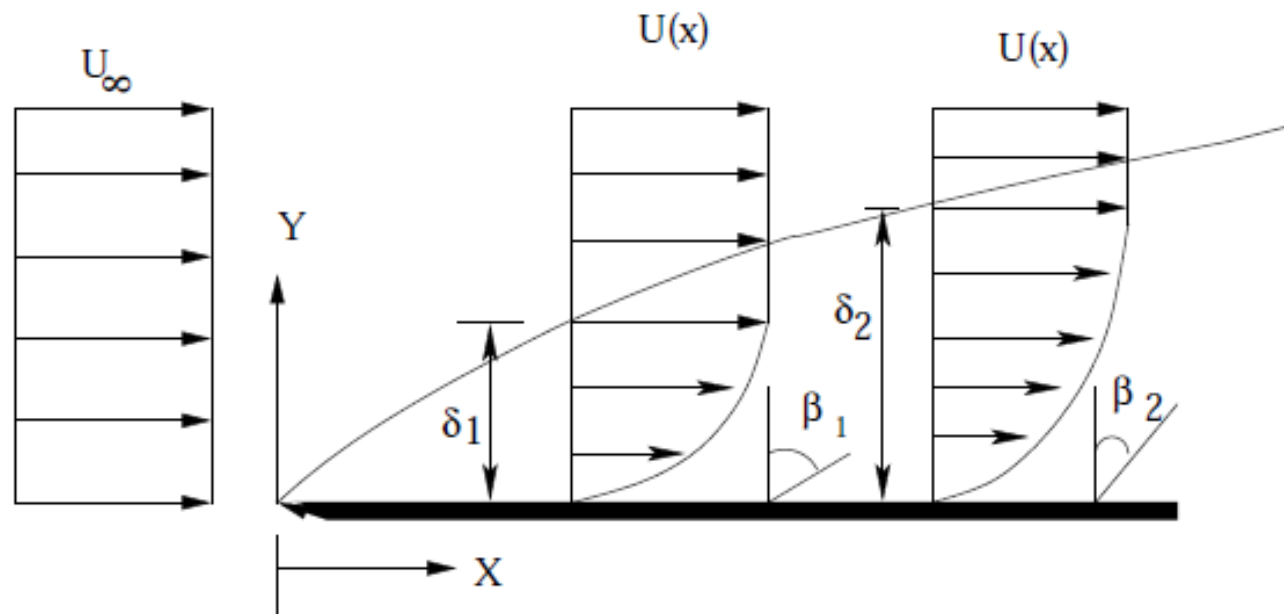


Figure 1.4: Change of Slope in the Downstream

Thermal Boundary Layer

Just as the velocity boundary layer, a thermal boundary layer should develop if the temperatures at the fluid free stream and the solid-surface differ. Fluid particles that come into contact with the plate achieve thermal equilibrium of the plate's surface temperature and the temperature gradient develops in the fluid. The region of the fluid in which these temperature gradients exist, is thermal boundary layer (Figure 1.5). We can write $\delta_T \equiv$ value of y for which, $(T_w - T)/(T_w - T_\infty) = 0.99$

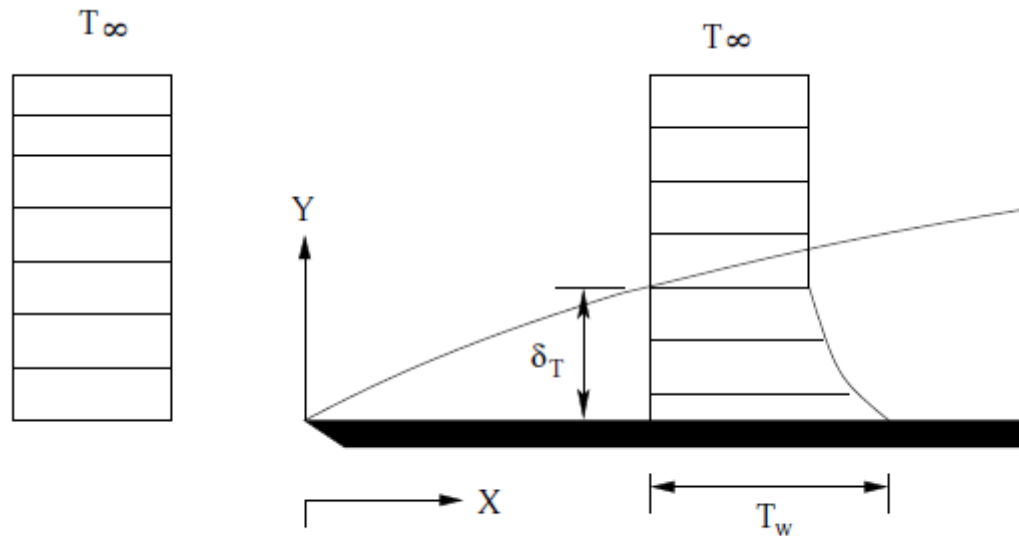


Figure 1.5: Thermal Boundary Layer on a Heated Plate

At any distance from leading edge, $\dot{q}_x = -k_f \frac{\partial T}{\partial y}|_{y=0}$, because at the surface, energy transfer is through conduction. The expression is exact because at the surface, there is no fluid motion. So, we can write

$$h(T_w - T_\infty) = -k_f \frac{\partial T}{\partial y}|_{y=0} \quad (1.90)$$

or,

$$h = \frac{-k_f \left(\frac{\partial T}{\partial y} \right) |_{y=0}}{(T_w - T_\infty)} \quad (1.91)$$

Hence, the conditions in the thermal boundary layer which strongly influence the wall temperature gradient $\left(\frac{\partial T}{\partial y} \right) |_{y=0}$, determine the rate of heat transfer across the boundary layer. Since $(T_w - T_\infty)$ is a constant (independent of x), and δ_T increases with increasing x , temperature gradient in the boundary layer must decrease with increasing x (Figure 1.6). This means $\frac{\partial T}{\partial y} |_{y=0}$ decreases with increase in x , whereby q_x and h also decrease with increase in x .

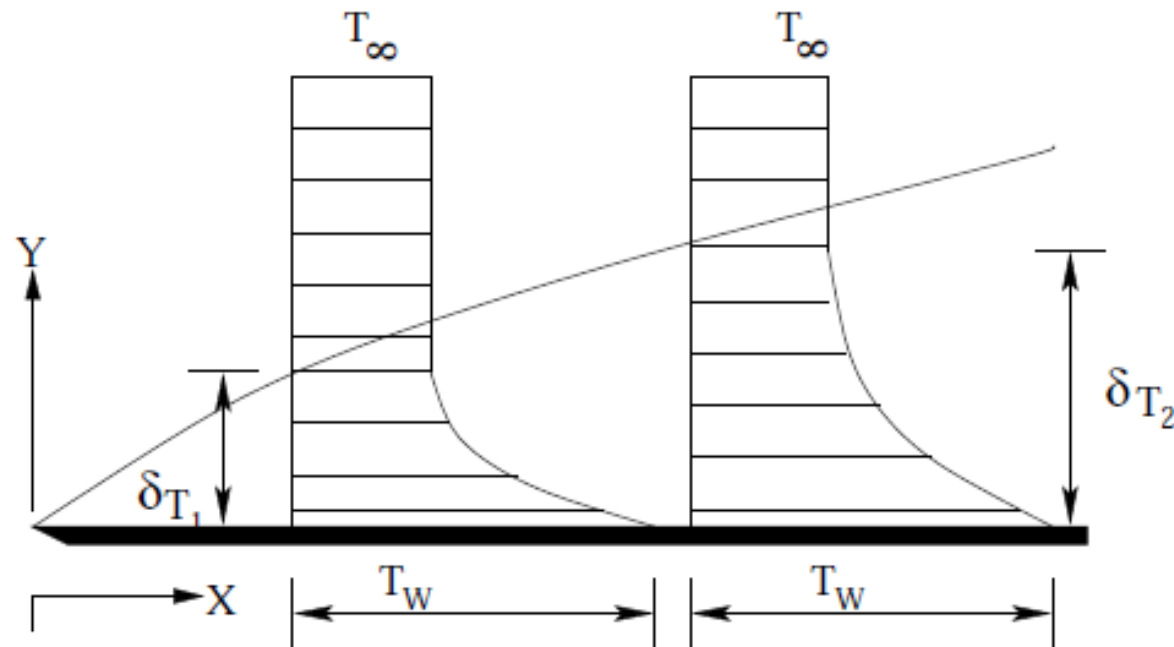


Figure 1.6: Change of Temperature Gradient within the Thermal Boundary Layer

More about Velocity Boundary Layer and Thermal Boundary Layer

Navier-Stokes equation along with the equation of continuity:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- In the region, very near to the wall, fluid motion is retarded until it adheres to the surface. The transition of main-stream velocity from zero at the surface to full magnitude takes place through the boundary layer. Its thickness is δ which is function of x .
- In order to find out velocity distribution in the field and velocity gradients at the wall, full Navier-Stokes equations should be solved. This is almost impossible analytically and can be solved only by numerical techniques.
- Easier approach would be to divide the flow field in two regions, - inviscid and viscous (boundary layer zone). The inviscid flow is irrotational in this case. Here we solve different governing equations in the inviscid zone and the boundary layer zone.
- To solve the governing equation in the boundary layer zone, boundary layer equation should be derived.

Flow over a flat plate (steady)

Let the dimensionless variables be defined as the following

$$u^* = u/U_\infty, v^* = v/U_\infty, p^* = p/\rho U_\infty^2, x^* = x/L, y^* = y/L$$

The equations become

$$\begin{aligned}u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} &= -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re} \left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \right] \\u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} &= -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re} \left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \right] \\\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} &= 0\end{aligned}$$

Following are the scales for the boundary layer variables:

Variable		dimensional scale		nondimensional scale
u	:	U_∞	:	1
x	:	L	:	1
y	:	δ	:	$\epsilon = (\delta/L); \epsilon \ll 1$

Consider, the continuity equation: $\partial u^*/\partial x^* \sim O(1) \Rightarrow \partial v^*/\partial y^*$ should be $O(1)$. [We are not allowed to drop any term from the continuity equation, we do not allow accumulation or annihilation of mass]. Now, v^* has to be of order ϵ because y^* at its maximum is ϵ ($\delta/L \ll 1$).

Take a recourse to the Navier-Stokes equations

x momentum

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re} \left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \right]$$
$$(1) \frac{(1)}{(1)} \quad (\epsilon) \frac{(1)}{(\epsilon)} = (1) \quad (\epsilon^2) \left[\frac{(1)}{(1)} \quad \frac{1}{(\epsilon^2)} \right]$$

y momentum

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re} \left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \right]$$
$$(1) \frac{(\epsilon)}{(1)} \quad (\epsilon) \frac{(\epsilon)}{(\epsilon)} = (?) \quad (\epsilon^2) \left[\frac{(\epsilon)}{(1)} \quad \frac{\epsilon}{(\epsilon^2)} \right]$$

Continuity

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$
$$\frac{(1)}{(1)} \quad \frac{(\epsilon)}{(\epsilon)}$$

Within the boundary layer $I_f \sim V_f$ (Inertia force $\approx$ Viscous force)

$$\rho \frac{U_\infty U_\infty}{L} \approx \mu \frac{U_\infty}{\delta^2}$$

or,

$$\frac{\rho U_\infty L}{\mu} \approx \frac{L^2}{\delta^2} \quad \Rightarrow \quad Re \approx \frac{1}{\epsilon^2}$$

Two outcomes $\Rightarrow$ x-momentum equation can be re-written based on order of magnitude approximation (retaining the terms of order 1).

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re} \left[\frac{\partial^2 u^*}{\partial y^{*2}} \right]$$

and

$$\frac{\partial p^*}{\partial y^*} = O(\epsilon)$$

The meaning of Equation (1.104) is the following. There is no variation in pressure in y direction within the boundary layer. Pressure is impressed on the boundary layer by the outer flow. The pressure, p is only a function of x within the boundary layer.

At the outer edge of the boundary layer if we substitute x-momentum equation, we shall obtain

$$u^* \frac{du^*}{dx^*} = -\frac{dp^*}{dx^*}$$

In dimensional form

$$u \frac{du}{dx} = -\frac{1}{\rho} \frac{dp}{dx}$$

or,

$$p + \frac{1}{2} \rho u^2 = \text{constant}$$

Boundary conditions:

$$\text{at } y = 0, u = 0 = v$$

$$\text{or, at } y^* = 0, u^* = 0 = v^*$$

$$\text{at } y = \delta, u = U_\infty(x)$$

$$\text{or, at } y^* = \epsilon, u^* = 1$$

Derivation of the equation for Thermal Boundary layer

$$Pr = \frac{\mu c_p}{k} = \frac{\mu}{\rho} \frac{\rho c_p}{k} = \frac{\nu}{\alpha} = \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}} \sim \left(\frac{\delta}{\delta_T} \right)$$

The energy equation, for the incompressible flows can be written as

$$\begin{aligned} u^* \frac{\partial \theta}{\partial x^*} + v^* \frac{\partial \theta}{\partial y^*} &= \frac{1}{Re \, Pr} \left[\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}} \right] \\ (1) \frac{(1)}{(1)} \quad (\epsilon) \frac{(1)}{(\epsilon)} &= \frac{(\epsilon^2)}{1} \left[\frac{(1)}{(1)} \quad \frac{1}{(\epsilon^2)} \right] \end{aligned}$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}$$

$$T_{min} = T_{\infty}, \quad T_{max} = T_w \quad \text{and} \quad 0 < \theta < 1$$

The Prandtl number varies in a wide range from value of order of 0.01 for liquid metals to a value of order of 1000 for viscous oils. Simplifications are possible for very small or very large Prandtl numbers. We shall avoid such simplifications in order to keep the boundary layer equations general. However, Re is always large in our consideration. We get finally

$$u^* \frac{\partial \theta}{\partial x^*} + v^* \frac{\partial \theta}{\partial y^*} = \frac{1}{Re \, Pr} \left[\frac{\partial^2 \theta}{\partial y^{*2}} \right] \quad (\text{nondimensional})$$

or,

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} \quad (\text{dimensional})$$

Physical significance is $\frac{\partial T}{\partial y} \gg \frac{\partial T}{\partial x}$, i.e., axial conduction in fluids much less than that of the transport (diffusion) rate in the normal direction. Please note that $\frac{\partial T}{\partial x}$ cannot be zero as long as heat transfer is taking place. However, $\frac{\partial T}{\partial y} \gg \frac{\partial T}{\partial x}$.

$Nu = (hL/k)$ = Nusselt number = Dimensionless temperature gradient at the surface

$Nu = Nu(u^*, v^*, x^*, y^*, \theta, Re, Pr)$ for forced convection in the absence of dissipation and volumetric heat generation.

This can also be shown that the non-dimensional form of the thermal boundary layer equation becomes

$$u^* \frac{\partial \theta}{\partial x^*} + v^* \frac{\partial \theta}{\partial y^*} = \frac{1}{RePr} \left[\frac{\partial^2 \theta}{\partial y^{*2}} \right] + \frac{Ec}{Re} \left[2 \left(\frac{\partial u^*}{\partial y^*} \right)^2 \right]$$

in the presence of viscous dissipation and the subsequent dimensional form can be written as

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left[\frac{\partial^2 T}{\partial y^2} \right] + \mu \left[2 \left(\frac{\partial u}{\partial y} \right)^2 \right]$$

In the above analysis, $\left(\frac{\partial u}{\partial y} \right)^2$ has been considered as the leading representative term of the viscous dissipation function. Obviously, for such flows,

$$Nu = Nu(u^*, v^*, x^*, y^*, \theta, Re, Pr, Ec)$$

Important Definitions

- Boundary layer thickness $\equiv \delta = 5.0x/\sqrt{Re_x}$ (for laminar flow past a flat plate)

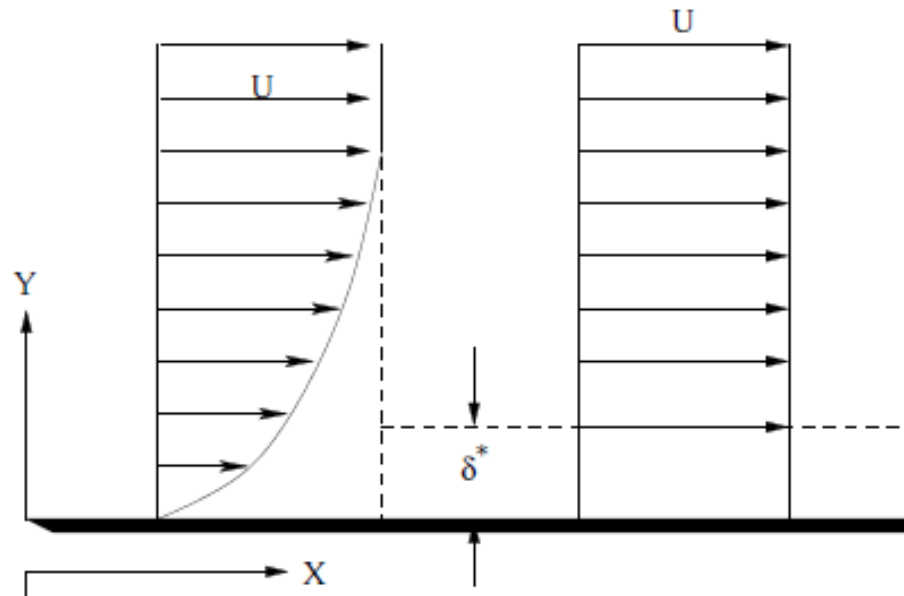


Figure 1.7: Displacement Thickness

- Displacement thickness $\equiv \delta^* =$ distance by which the external potential flow is displaced outwards as a consequence of decrease in velocity in the boundary layer (Figure 1.7).

$$U\delta^* = \int_0^{\infty} (U - u)dy$$

$$\delta^* = \int_0^{\infty} \left(1 - \frac{u}{U}\right) dy$$

- Momentum thickness, δ^{**} , is defined as the loss of momentum in the boundary layer as compared with that of potential flow

$$\rho U^2 \delta^{**} = \int_0^\infty \rho u (U - u) dy$$

$$\delta^{**} = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U} dy$$

- Laminar Flow : Fluid motion is well ordered, layers slide over each other.
- Turbulent Flow: Fluid motion is highly irregular and characterized by velocity fluctuations (Figure 1.8). The critical Reynolds number is defined as, $Re_c = \rho U x_c / \mu \approx 5 \times 10^5$ (for external flows)

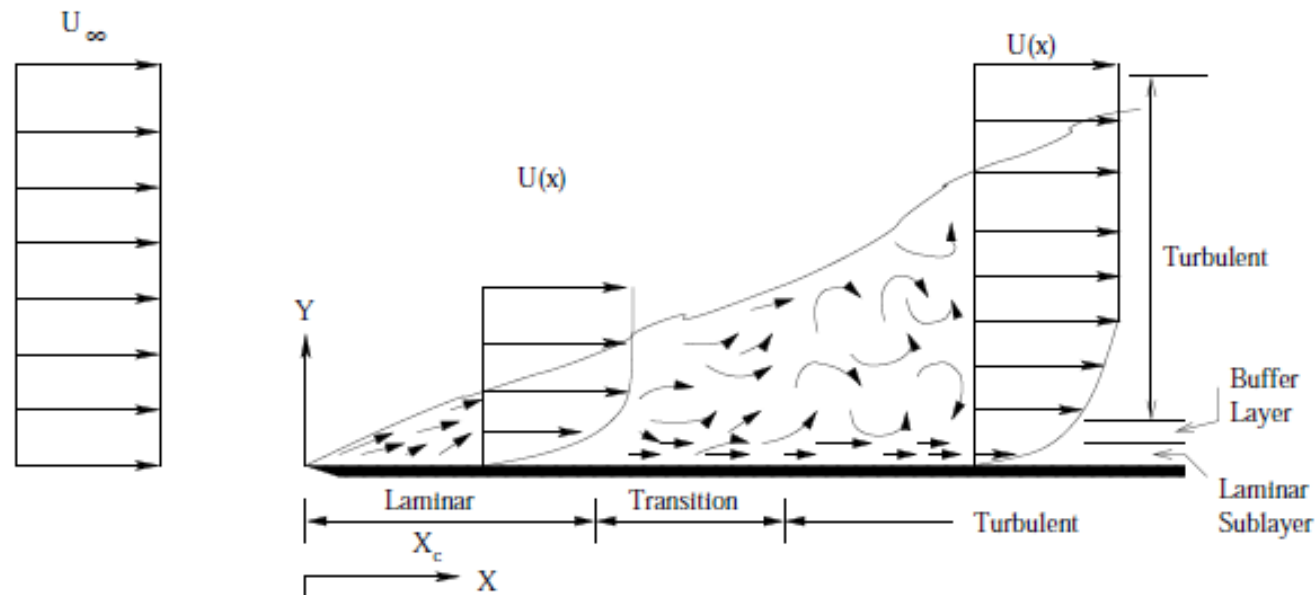


Figure 1.8: Transition in a Flat Plate Flow

Prandtl number and Ratio of Boundary Layers

Let us consider a case for $\delta_T > \delta$ (see Figure 1.9). From the thermal boundary layer consideration, a scale analysis within the boundary layer gives

$$\begin{aligned} u \frac{\partial T}{\partial x} &\sim \alpha \frac{\partial^2 T}{\partial y^2} \\ \text{or, } U_\infty \frac{\Delta T}{L} &\sim \alpha \frac{\Delta T}{(\delta_T)^2} \\ \text{or, } \left(\frac{\delta_T}{L} \right)^2 &\sim \frac{\alpha}{U_\infty L} \\ \text{or, } \left(\frac{\delta_T}{L} \right)^2 &\sim \frac{\nu}{U_\infty L} \cdot \frac{\alpha}{\nu} \end{aligned}$$

From the velocity boundary layer consideration, a scale analysis within the boundary layer gives

$$\begin{aligned} u \frac{\partial u}{\partial x} &\sim \nu \frac{\partial^2 u}{\partial y^2} \\ \text{or, } U_\infty \frac{U_\infty}{L} &\sim \nu \frac{U_\infty}{\delta^2} \\ \text{or, } \left(\frac{\delta}{L} \right)^2 &\sim \frac{\nu}{U_\infty L} \end{aligned}$$

From equations (1.120) and (1.121) we get

$$\begin{aligned}\left(\frac{\delta_T}{L}\right)^2 &\sim \left(\frac{\delta}{L}\right)^2 \cdot \frac{\alpha}{\nu} \\ \text{or, } \left(\frac{\delta}{\delta_T}\right)^2 &\sim Pr \\ \text{or, } (Pr)^{\frac{1}{2}} &\sim \left(\frac{\delta}{\delta_T}\right)\end{aligned}$$

Through a similar analysis, it can be shown that

$$(Pr)^{\frac{1}{3}} \sim \left(\frac{\delta}{\delta_T}\right) \quad \text{for } \delta > \delta_T$$

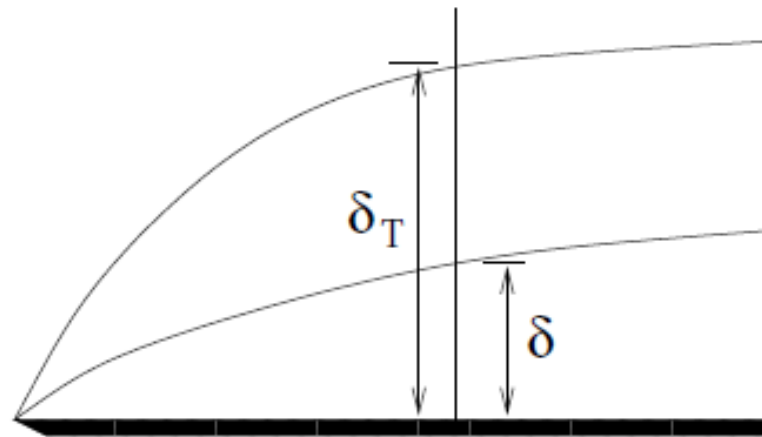


Figure 1.9: The situation $\delta_T > \delta$



Thank you